Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

number phase from objects with which he is familiar. In fingers and apples he has an interest and any new ideas about these old friends are seized with some avidity.

To question the child's power to abstract the difference of like objects is to question his power of imagination. that prime requisite of a mathematician which power the child has most preeminently.

For the further work in arithmetic this development means the constant and recurring emphasis of the decimal system. Eighteen hundred years ago Nikomachus of Gerasa, the father of Arithmetic, did better than we in giving a table of multiples only up to 9×9, and the earliest printed arithmetics used the ordinary multiplication table only to 9×9 or 10×10. By so doing is emphasized the use of the decimal system, as all further multiplicative combinations involve only these fundamental number of facts. With the addition tables, too, there needs to be emphasis on the system. Here we may see one great way to simplify our arithmetic, namely, to play the system.

◆

A SOLUTION OF THE BIQUADRATIC EQUATION.

By **LEROY A. HOWLAND, Middletown, Connecticut.**

The following solution of the biquadratic rests upon the analytic operations suggested by geometrical considerations. No assumption is made as to the form of the solution. Each step in the solution and in the discussion of multiple roots has its geometrical analogon. The cubic resolvent has a geometrical meaning and its discriminant is at the same time the discriminant of the biquadratic. M. Fritz Hofmann has used the degenerate members of a family of conics through four points to determine the roots of the biquadratic,* but in an entirely different way. He does not appear to have extended his method to a discussion of the nature of the roots.

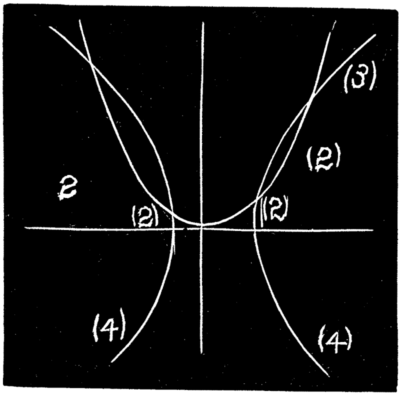


Fig. 1.

1. Solution. Let the biquadratic be

$$a_0z^4+4a_1z^3+6a_2z^2+4a_3z+a_4=0 \qquad (1)$$

$$(a\neq0)$$

This goes over by the substitution $z=\frac{x-a_1}{a_0}$ into

$$x^4+ax^2+bx+c=0, \qquad (2)$$

---

* *Nouvelles Annales de Mathématiques, Troisième Série, 7.*

where $a=6(a_0a_2-a_1^2)$,

$$b=4(a_3^2-3a_0a_1a_2+2a_1^3),$$

$$c=a_0^3a_4-4a_0^2a_1a_3+6a_0a_1^2a_2-3a_1^4.$$

(2) represents four straight lines parallel to the Y axis. These cut the parabola

$$y=x^2 \tag{3}$$

in four finite points. The conic

$$ax^2+y^2+bx+c=0 \tag{4}$$

cuts (3) in these same four points (cf. Fig. 1). A conic through the intersection of (3) and (4) is

$$(a-\lambda)x^2+y^2+bx+\lambda y+c=0. \tag{5}$$

This degenerates if

$$-D=\begin{vmatrix} a-\lambda & 0 & b/2 \\ 0 & 1 & \lambda/2 \\ b/2 & \lambda/2 & c \end{vmatrix}=0$$

or, (6) $$\lambda^3-a\lambda^2-4c\lambda+4ac-b^2=0.$$

The solution of (6) gives three pairs of straight lines, each of which intersects the others in the intersections of (2) and (3). The solution of the biquadratic is then accomplished by the solution of (6) and certain pairs of simultaneous *linear* equations (cf. Fig. 2).

Solving (5) for y, we have

$$2y=-\lambda\pm\sqrt{[4(\lambda-a)x^2-4bx+\lambda^2-4c]}.$$

Using (6), this gives

$$2y=-\lambda\pm[2\sqrt{(\lambda-a)}x-\frac{b}{\sqrt{(\lambda-a)}}].$$

If λ_j and λ_k are two roots of (6) we have the following values for x,

$$
\begin{aligned}
&\tfrac{1}{2}\left[\pm\frac{\lambda_j-\lambda_k}{\mu_i+\mu_k}+\frac{b}{\mu_j\,\mu_k}\right]\\
&\tfrac{1}{2}\left[\pm\frac{\lambda_j-\lambda_k}{\mu_j-\mu_k}-\frac{b}{\mu_j\,\mu_k}\right]
\end{aligned}
\tag{7}
$$

where $\mu_i=\sqrt{}(\lambda_i-a)$.

2. Multiple Roots.

Fig. 2 represents the general case. Fig. 3 shows the special cases which may occur.

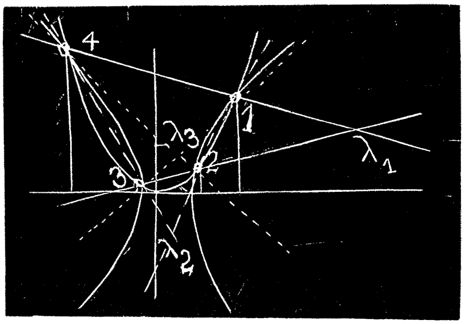


Fig. 2.

In cases I and II, two pairs of lines become coincident. Hence (6) must have a double root.

In case III, since the points approach coincidence along the parabola, each of the lines 12, 23, and 31 approaches the tangent to the parabola at the point of coincidence. Hence equation (6) must have a triple root.

Similar argument shows that in case IV all six lines coincide. We shall now express these conditions analytically.

I. The condition for a double root of (6) is

$$
\begin{aligned}
&G^2+4H^3=0\\
\text{where } &G=\tfrac{1}{27}(72ac-2a^3-27b^2),\\
&H=-\tfrac{1}{9}(a^2+12c),
\end{aligned}
$$

or in terms of the coefficients of (1),

$$
\begin{aligned}
G&=16a_0^3(a_0a_2a_4-a_0a_3^2+2a_1a_2a_3-a_1^2a_4-a_2^3),\\
H&=-4/3a_0^2(a_0a_4-4a_1a_3+3a_2^2).
\end{aligned}
$$

The quantities in parentheses are denoted by J and I, respectively. Hence, $G^2+4H^3=-256/27a_0^3(I^3-27J^2)$, and the necessary and sufficient condition for a double root of (1) is

$$\triangle\equiv I^3-27J^2=0. \tag{8}$$

II. For two double roots, the lines corresponding to the double root of (6) must be coincident. (5) will represent parallel lines if $D=0$ and $\lambda-a=0$.

We see by substitution that a necessary condition that $\lambda=a$ be a solution of (6) is

$$b\equiv a_0^2a_3-3a_0a_1a_2+2a_1^3=0. \tag{9}$$

Equation (5) then becomes

$$y^2+ay+c=0,$$

and we obtain the further necessary condition

$$a^2-4c=0. \tag{10}$$

These conditions, (9) and (10), are evidently also sufficient.*

III. A triple root of (6) is necessary and sufficient for a triple root of (1). The condition for this is $G=H=0$, or what is the same,

$$I=J=0. \tag{11}$$

IV. For a quadruple root we must have, in addition to (11), the conics degenerate into coincident lines. This gives, as before, $\lambda=a$, the triple root of (6), and hence $b=0$. It then follows that $a=c=0$. These conditions are sufficient, for if they are fulfilled, (2) becomes $x^4=0$, and (1) becomes

$$\frac{1}{a_0^3}(a_0z+a_1)^4=0.$$

In terms of the coefficients of (1) the conditions may be written

$$\frac{a_0}{a_1}=\frac{a_1}{a_2}=\frac{a_2}{a_3}=\frac{a_3}{a_4} \tag{12}$$

unless one of the quantities a_1, a_2, a_3, a_4 is zero, in which case they are all zero and (1) reduces to $a_0z^4=0$.

3. Reality of Roots.

I. If $G^2-4H^3>0$ or $\Delta<0$, the equation (6) has two imaginary roots, $\alpha\pm i\beta$.

$$\mu_j=\sqrt{(\alpha-a+i\beta)},\quad \mu_k=\sqrt{(\alpha-a-i\beta)},$$
$$(\mu_j\,\mu_k)^2=(\alpha-a)^2+\beta^2,$$
$$(\mu_j\pm\mu_k)^2=2(\alpha-a)\pm2\sqrt{[(\alpha-a)^2+\beta^2]},$$
$$\lambda_j-\lambda_k=2i\beta.$$

---

* We find the statement sometimes made that (9) and (10) along with the condition $\Delta=0$ are equivalent to two independent conditions only. If this means merely that $\Delta=0$ is a consequence of (9) and (10) the statement is correct. It is not true that $\Delta=0$ with either (9) or (10) forms a set of sufficient conditions for two double roots.

Hence it appears that $\mu_j \mu_k$ and $\mu_j+\mu_k$ are real, $\mu_j-\mu_k$ and $\lambda_j-\lambda_k$ are pure imaginary, and two values of x are real and two complex.

II. Before proceeding to the general cases $\triangle \gtreqless 0$, we shall take up the special case where $b=0$. (2) becomes $x^4+ax^2+c=0$, whence $x=\pm\sqrt{\frac{-a \pm \sqrt{(a^2-4c)}}{2}}$, $\triangle$ becomes $\frac{c(a^2-4c)^2}{16a_0^3}$, and we have two sub-cases.

1) $\triangle > 0$, hence $c>0$.

In order for x to be real the radicand above must be real and positive. For this it is necessary and sufficient that $a^2-4c>0$ and $a<0$. All values of x are then real, otherwise they are all complex.

2) $\triangle = 0$, hence either, a) $c=0$, or b) $a^2-4c=0$.

For a), (2) becomes $x^4+ax^2=0$, and has a real double root and two other roots which are real or complex according as $a \leqq 0$ or $a>0$.
For b), (2) becomes $(x^2+\frac{1}{2}a)^2=0$, and has two double roots, both real or both complex under the same conditions as in a.

III. We shall now assume $b \neq 0$ and consider the otherwise general case $\triangle > 0$.

If $G^2+4H^3<0$ or $\triangle > 0$ the roots of (6) are all real. In this case it is evident from the form of x in (7) that all values of x are real if all roots of (6) are greater than a and all complex if two roots of (6) are less than a. We will now apply the theorem of Fourier to see where the roots of (6) lie.

$$\text{Let } f(\lambda)=\lambda^3-a\lambda^2-4c\lambda+4ac-b^2,$$
$$f'(\lambda)=3\lambda^2-2a\lambda-4c,$$
$$f''(\lambda)=6\lambda-2a,$$
$$f'''(\lambda)=6.$$

Whence $f(a)=-b^2$, $f'(a)=a^2-4c$, $f''(a)=4a$, $f'''(a)=6$. This suggests the following division:

1) $a^2-4c>0 \quad a>0$,
2) $a^2-4c>0 \quad a<0$,
3) $a^2-4c<0 \quad a>0$,
4) $a^2-4c<0 \quad a<0$,
5) $a^2-4c<0 \quad a=0$,
6) $a^2-4c=0 \quad a>0$.

The other possibilities are $a^2-4c>0$, $a=0$; $a^2-4c=0$, $a<0$; $a^2-4c=0$, $a=0$;

but these are all ruled out, for G^2+4H^3 reduces to $b^4-\frac{64}{27}c^3$; $b^2(b^2-\frac{32}{27}a^3)$; b^4 in the three cases, respectively, and can in no case be negative. We have the following table:

	$\lambda=a$						$\lambda=\infty$	$\lambda=-\infty$
	1	2	3	4	5	6		
$f(\lambda)$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$-$
$f'(\lambda)$	$+$	$+$	$-$	$-$	$-$	0	$+$	$+$
$f''(\lambda)$	$+$	$-$	$+$	$-$	0	$+$	$+$	$-$
$f'''(\lambda)$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$

It appears then in case (2) that all roots are greater than a, while in every other case only one root is greater than a. In the case of real distinct roots of (6) then the necessary and sufficient condition that all values of x be real is $a<0$, $a^2-4c>0$. If these conditions are not fulfilled the values of x are all complex.

IV. $\triangle=0$, with the further assumption that $b\neq 0$. Let $\lambda_j=\lambda$ be the simple root of (6). It is greater than a and in case the double root is also greater than a, all values of x are real. The condition for this is easily seen to be as in case III, $a<0$, $a^2>4c$. Putting $2x$ in the form

$$\pm\sqrt{(\lambda-a)}+\frac{\mp\sqrt{(\lambda-a)}\,(\lambda_k-a)+b}{\sqrt{(\lambda-a)}\sqrt{(\lambda_k-a)}},\quad \pm\sqrt{(\lambda-a)}+\frac{\pm\sqrt{(\lambda-a)}\,(\lambda_k-a)-b}{\sqrt{(\lambda-a)}\sqrt{(\lambda_k-a)}},$$

we see that in all other cases, since $\lambda-a>0$ while $\lambda_k-a<0$, we can have real values of x only when

$$\sqrt{(\lambda-a)}\,(\lambda_k-a)=\pm b.$$

Since the sum of the three roots of (6) is a, we have $\lambda_k=\frac{a-\lambda}{2}$, and we can transform the condition above into

$$(13)\qquad \begin{cases} (\lambda+a)\sqrt{(\lambda-a)}=+2b, \text{ or} \\ (\lambda+a)\sqrt{(\lambda-a)}=-2b. \end{cases}$$

We can have real values of x then only if

$$(14)\qquad (\lambda+a)^2(\lambda-a)=-4b^2.$$

The resultant of (6) and (14) is $\triangle$. This is, of course, to be foreseen, since whenever $\triangle=0$ there are two equal values and hence always two real values.

(The case where there are two double roots has been excluded because $b \neq 0$.) The conditions (13) are both satisfied only in case $b=0$, and this we have already considered.

We will summarize the results of this section.

1) $\triangle < 0$, two roots real, two complex.

2) $\triangle > 0$,

   a) $a<0$, $a^2-4c>0$, all roots real.

   b) Otherwise all roots complex.

3) $\triangle = 0$,*

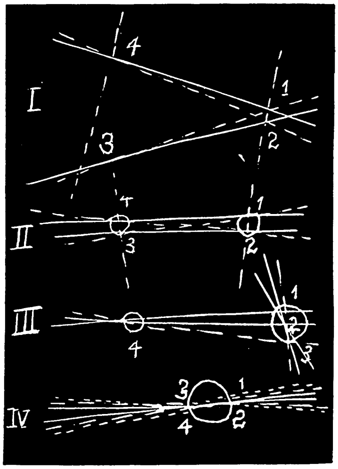
   a) $a<0$, $a^2-4c>0$, all roots real.

   b) Otherwise two real and two complex, except in the following cases:

   c) $a>0$, $b=0$, $a^2-4c=0$, two double complex roots.

   d) $a<0$, $b=0$, $a^2-4c=0$, two double real roots.

   e) $a=b=c=0$, all roots real.



Double root.

2 Double roots.

Triple root.

Quadruple root.

Fig. 3.

---

* In many standard works this case is not considered. It is discussed by Heinrich Weber in his *Lehrbuch der Algebra*, but the discussion contains an error. [Weber noticed this error himself in his *Berichtigungen* at the end of the volume.] One finds on page 278 of volume I, second edition, the statement: Die biquadratische Gleichung hat zwei Paare gleicher, reeller Wurzeln, wenn $D=0$, $a<0$, $a^2-4c=0$ und zwei Paare gleicher, imaginarer Wurzeln wenn $D=0$, $a>0$, $a^2-4c=0$.

The equation $x^4+6x^2+16x+9 \equiv (x+1)^2(x^2-2x-9)=0$ satisfies the second set of conditions above but does not have two pairs of equal roots nor are the roots all complex. The error is the same as that indicated in second note. Weber's first statement is correct, because for $a^2-4c=0$, D or $\triangle$ becomes, except for a constant factor, $b^2(27b^2-32a^3)$ and when a is negative this can vanish only when b is zero. Such is not the case however when a is positive.